ACTS OF THE INTERNATIONAL SYMPOSIUM
ON IBN TURK, KHWÂREZMÎ, FÂRÂBÎ, BEYRÛNÎ
AND IBN SÎNÂ
(ANKARA 9-12 SEPTEMBER 1985)

JENS HØYRUP

Algebraic Traditions Behind Ibn Turk and Al-Khwârizmî

Offprint

ANKARA 1990

ALGEBRAIC TRADITIONS BEHIND IBN TURK AND AL-KHWÂRIZMÎ*

JENS HØYRUP**

I. THE TRADITIONAL STATE OF THE PROBLEM

Since the discovery some fifty years ago that certain cuneiform texts solve equations of the second degree[1] the idea has been close at hand that the early Islamic algebra known from Al-Khwârizmî and his contemporary Ibn Turk continues and systematizes an age-old tradition. More recently, Anbouba (1978, 76ff) has also made it clear that the two scholars worked on a richer contemporary background than can be seen directly from their extant works. In fact, the same richer tradition can be glimpsed e.g. from some scattered remarks in Abû Kâmil's *Algebra*—cf. below, section V.

Hitherto, a main argument for the assumption of continuity has been a reading of the Babylonian texts as descriptions of purely numerical algorithms, analogous to the rules given by Al-Khwârizmî. To exemplify the similarity, we may first look at Al-Khwârizmî's rule for the case "Roots and Squares are equal to Numbers", illustrated by "one square, and ten roots of the same, amount to thirty-nine dirhems":

> You halve the number of the roots, which in the present instance yields five. This you multiply by itself: the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the

[1] I use the term "Islamic" in the sense of "belonging to the culture and society of (Medieval) Islâm". In this sense, Thâbit as well as the young al-Samawᶜal are "Islamic" mathematicians, although they were not Muslims. I have chosen the term instead of the alternative "Arabic mathematics" because I consider Islâm and not the Arabic language the unifying force of the culture in question — cf. my (1984, esp. p. 29f).

roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine.

(Rosen 1831, 8).

A similar Babylonian problem (BM 13901 N°1) was translated as follows by Thureau-Dangin (I replace the sexagesimal numbers by current notation):

> J'ai additionné la surface et le côté de mon carré: 3/4. Tu poseras 1, l'unité. Tu fractionneras en deux: 1/2. Tu croiseras 1/2 et 1/2: 1/4. Tu ajouteras 1/4 à 3/4: 1. C'est le carré de 1. Tu soustrairas 1/2, que tu as croisé de 1: 1/2, le côté du carré.

The styles of the two treatments appear indeed to be quite similar. The tradition seems to be one of *correct but unjustified and unexplained numerical computation,* and a main innovation of the two Islamic algebrists appears to be their introduction of "naive-geometric" justifications for the traditional standard procedures.

In terms which I shall use recurrently below, it looks from the traditional translations as represented by my extract from TMB as if the *basic conceptualization* —i.e. the ontological status given to the fundamental entities used to represent the various concrete quantities dealt with in real or faked practical problems (be it numbers found in the tables of reciprocals, areas of fields, or prices) —was arithmetical: The "area" and the "side" of the square are, in this traditional interpretation, nothing but names indicating the artihmetical relations between the powers of an unknown number, as it is the case in Diophantos' *Arithmetica.* Similarly, the *procedure* seems to be arithmetical— as it is also the case in Diophantos and in normal Islamic and Western "rhetorical" algebra, (In contrast, Al-Khwârizmî's and Ibn Turk's above-mentioned justifications are geometrical according to their *procedure,* although the *conceptualization* is arithmetical even here, the square and its side being thought to *represent* the numbers *mâl* and *jadr,* "wealth" and "root", i.e. unknown and its square root).

II. A NEW INTERPRETATION OF OLD BABYLONIAN ALGEBRA

The above scenario for the development from Babylonian to early Islamic algebra is challenged by the results of a close investigation of the procedures and the basic conceptualization of Old Babylonian algebra in which

I have been engaged for some years. [2,3] Close attention to the structure and use of the terminology shows, together with various other considerations, that the traditional reading of the texts provides us with a mathematically homomorphous but not with a correct picture: The lengths and areas of the texts have to be accepted at face value, in agreement with a geometric conceptualization. Similarly, the procedure turns out to be one of "naive", constructive geometry of areas, very similar to but more primitive than the justifications found in Al-Khwârizmî and Ibn Turk.

In order to support these statements I shall translate and explain three Old Babylonian problems, using the more precise meaning of terms which have come out of my investigation.

Let us first have a second look at the text quoted above from Thureau-Dangin (BM 13901 translated this time from the transliterated text in MKT III, 1):

> The surface and the square-line I have accumulated: 3/4. 1 the projection you put down. The half of 1 you break, 1/2 and 1/2 you make span (a rectangle, here a square), 1/4 to 3/4 you append: 1, makes 1 equilateral. 1/2 which you made span you tear out inside 1: 1/2 the square-line.

The terminology is awkward, and must be so in order to render if only imperfectly a structure of concepts and operations different from ours. The "square-line" *(mitḫartum)* designates a square identified by (and hence with)the length of its side (as we have identified the figure with its area since the Greeks). The term means "that which confronts (its equivalent)" and derives from *maḫārum,* a word which is close to Arabic *qabila* in its total range of connotations. You "append" *(waṣābum)* X to Y when performing a concrete (not abstract-arithmetical) addition in which the entity X conserves its identity (as a capital conserves its identity even when the bank

---

[2] First briefly communicated (in Danish) in my (1982). Later preliminary presentation in my (1984a), revised as (1985). MS in progress (1985a).

[3] In the first instance, I speak only of the Old Babylonian algebra texts, dating from c. 1800 B.C. to c. 1600 B. C. In section III I shall return to the question of the next documented phase of Babylonian algebra, the Seleucid texts (3d to 2nd century B. C.).

adds the interests of the year), while you "accumulate" *(kamārum)* them in a more abstract addition where both addends loose their identity (apparently, the "accumulation" designates a real addition of measuring numbers). The "projection" *Wāṣītum)* is the width 1 which from a line of length X makes a rectangle of area $X \cdot 1 = X$. To "put down" translates *šakānum,* an all-purpose-term close to English "to put" or "to place" or to Arabic *waḍaʿa.* To "break" *(ḥepûm)* is used with such bisections which have a geometric meaning (but not for general division by 2) Two lines are "made span" *(šutākulum)* when a rectangle is created ("built" is the Babylonian expression *banûm,* cf. Arabic *banâ).* The "equilateral" is another (Sumerianizing) term for the quadratic figure (a verb meaning "to be equal"), and the phrase "x makes y equilateral" is used to tell that y is the side of a square of area X. To "tear out" *(nasāḥum)* is a process of concrete, identity-conserving subtraction, the inverse of "appending".

With these explanations in mind one should be able to follow the procedure on Figure 1. Firstly the "projection" is placed projecting from one of the sides of the square. Next it is "broken" (together with the whole appurtenant rectangle), and the outer part moved so that the two "span" a square (dashed line in the third step) of area $1/2 \cdot 1/2 = 1/4$, which is appended to the gnomon resulting from the displacement of the broken-off rectangle. This larger square then has an area $3/4 + 3/4 = 1$, and hence a side 1. The broken-off and displaced $1/2$, which is part of this side 1, is "torn out" from it, leaving back the required "square-line".

If we compare this with Al-Khwârizmî's second justification of the case · "a Square and ten Roots are equal to thirty-nine Dirhems" (Rosen 1831, 15f), we find a very close agreement. Problem N° 23 of the same Old Babylonian tablet provides us with a parallel to his first variant of the justification, where $10 \cdot x$ are distributed equally along the four edges of the square $x \cdot x$ (MKT III, 4 f; translated with sexagesimal numbers):

> The surface of the four fronts and the surface I have accumulated: 0; 40, 41. 4, the four fronts, you inscribe. The reciprocal of 4, 0;15. 0;15 to 0;41,40 you raise: 0;10,25 you inscribe. 1 the projection you append: 1; 10,25 makes 1;5 equilateral. 1 the projection which you appended you tear out: 0;5 you double until twice: 0;10 nindan confronts itself.

The translation calls for a few extra commentaries. To "raise" *(našûm)* is a term used when a concrete magnitude is to be calculated by multiplication. To "double" *(eṣēpum)* involves repetition two or eventually more times. The nindan is the basic unit of length (of value c. 6 m). Apart from single words, finally, it shall be emphasized that the grammatical construction used in the beginning makes it indubitably clear that *the* four fronts and not just 4 times the side are meant.

Let us now follow the text on Figure 2. The "surface of the four fronts" and the "projection" further down makes it clear that we have to begin with a cross-form configuration, as shown at the top. The multiplication by 1/4 (=0;15) is shown next: One fourth of the cross is considered alone. The square on the "projection" (identified as a geometric picture with its side, the "projection" itself) is "appended", transforming the gnomon into a square, the area of which is found to be 1;10,25. Hence the side (that which 1;10,25 "makes equilateral") must be 1;5. This side was composed by "appending" the "projection" to half the front; so, the "appended" 1 is torn out, and the remaining 0;5 is doubled (repeated concretely, not "raised to 2"), which gives us that front which "confronted itself" in the original square.
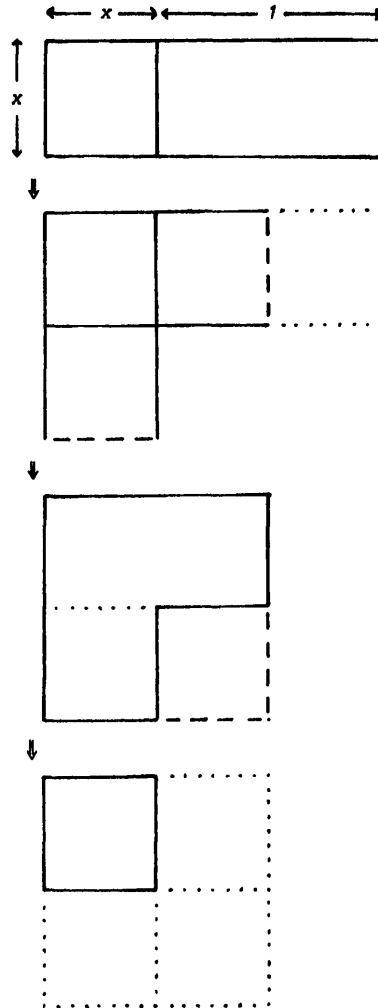
Figure 1. The geoetrical interpretation of BM 13901 N° 1. Cf. Ibn Turk's figure in Sayılı 1962, p. 163, and Al-Khwârizmî's in Rosen 1931, p. 16.

A third problem (AO 8862, N° 1; in MKT I, 108f) is more complicated. For easy reference, I divide it into sections.

A Length, width. Length and width I have made span: A surface I have built. I turn around. So much as that by which the length exceeds the width I have append ed to the inside of the surface: 183. I turn back. Length and width accumulated: 27. Length, width and surface how much?

B

27	183	accumulation
15	length	180 surface
12	width	

C You, by your making, append 27, the accumulation of length and width, to the inside of 183: 210. Append 2 to 27: 29.

D Half of it, that of 29, you break: 14 1/2. < 14 1/2 and 14 1/2 you make span > 14 1/2 times 14 1/2, 210 1/4. From the inside of 210 1/4 you tear out 210: 1/4 the remainder. 1/4 makes 1/2 equilateral. Append 1/2 to the first 14 1/2: 15 the length. You tear out 1/2 from the second 14 1/2: 14 the width.

E 2 which you have appended to 27 you tear out from 14, the width: 12, the true width.

F 15 the length, 12 the width make span: 15 times 12, 180 the surface. By how much does 15, the length, exceed 12, the width: It exceeds by 3; append it to the inside of 180, the surface, 183 the surface.

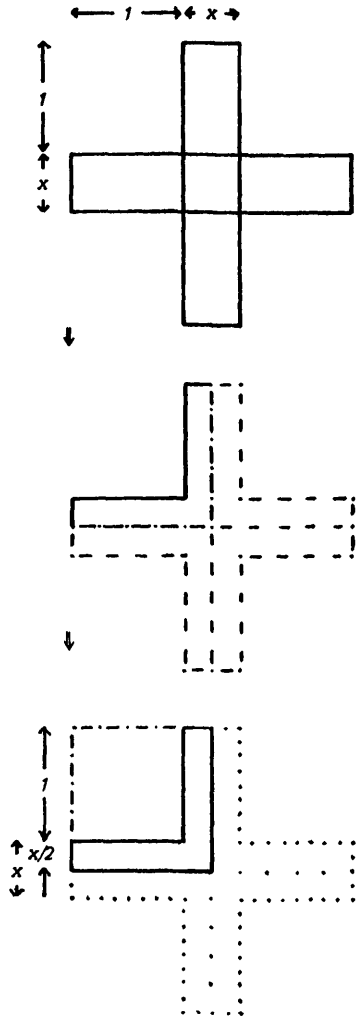The "length, width" in the beginning tell that the problem deals with a rec-

Figure 2. The geometrical interpretation of BM 13901, Nº 23. Cf. Al-Khwârizmî's figure in Rosen 1831, p. 15.

tangle. The "turning around" and "turning back" in A mark sections of the statement. B tells in advance the dimensions of the figure (and so, the procedure-part tells the student *how to obtain* these results known in advance). The "times" of D (and F) translates a - r à, the multiplicatory term of the multiplication tables (meaning literally "steps of"). The insertion > in D is made on the faith of parallel passages (among which one in F).

We may now follow the text on Figure 3. In the first section of the procedure (C), the known sum of length (1) and width (w) is "appended" "to the inside of" 183, yielding (when the onedimensional lengths are provided with an implicit "projection") a rectangle of length 1= 15, width W = w+2 = 14, and area 210 (α).

Through this geometric "change of variable" the problem is reduced to one of the standard problems of Babylonian algebra, which is solved in section D: The sum of length and width is bisected (β), and its halves are "made span" a square of area 14 1/2 · 14 1/2 = 210 1/4 (γ). The full-drawn gnomon inside this square, which is equal to the rectangle and hence to 210, is "torn out", leaving the small square (lower right corner) of area 1/4 and hence of side 1/2. Finally, this 1/2 is "appended" to the horizontal side of the large square, yielding the length 1 of the rectangle, and "torn out" from its vertical side, yielding its width w (δ) the width, that is, of the augmented rectangle.
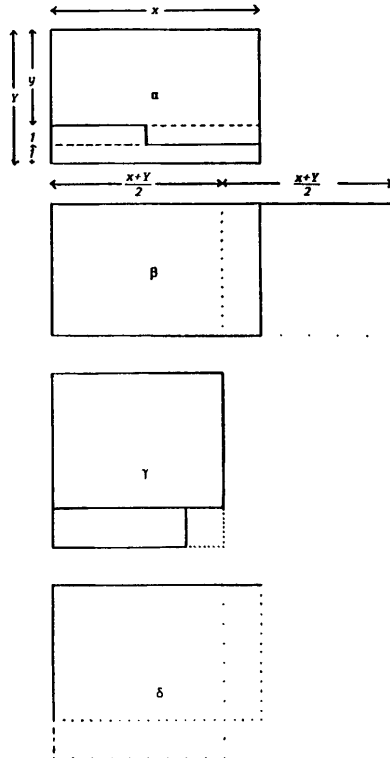

Figure 3. The geometrical interpretation of AO 8862 N° 1. Distortaged proportions. Cf. Ibn Turk's dentical figure in Sayılı 1962, p. 164, and Al-Khwârizmî's in Rosen 1831, p. 18.

In section E, the original ("true") width w is found by subtraction. Section F, finally, controls the correctness of the results.

By comparison with Al-Khwârizmî's *Algebra* ona finds that the procedure of section D is exactly the one given there to justify the algorithm for the case "a Square and twenty-one Dirhems are equal to ten Roots" (Rosen 1831, pp. 16-18). The same procedure is given by Ibn Turk (Saylı 1962, 163f), while Abû Kâmil uses a slightly different figure apparently inspired by *Elements* II. 5 (Levey 1966, pp. 44-46) — the proposition, indeed, to which Thâbit refers in *his* demonstration of the same matter (Luckey 1941, 106f).

III. SELEUCID TESTIMONY

As stated above, the next phase of documented Babylonian algebra belongs in the Seleucid era. I shall indicate the style of this phase by translating a simple problem (BM 34568 N° 9; translated after the text in MKT III, 15, as corrected in von Soden 1964, 48a):

Length and width accumulated: 14, and 48 the surface. I do not know the name. 14 times 14, 196. 48 times 4, 192. Go up from 192 to 196: 4 remains. How much times how much shall I go in order to get 4: 2 times 2, 4. Go up from 2 to 14, 12 remains. 12 times 1/2, 6. 6 the width. Add 2 to 6, 8 the length.

Figure 4. A geometric figure wihich will serve as "naive" justification for the calculational steps in BM 34568 N° 9 and which, when diagonals are drawn in the rectangles, will demonstrate the Pytagorean theorem and a variety of derived identities through simple counting.

The most conspicuous change is probably the completely arithmetical conceptualization of a problem which is formally presented as geometric. Numbers in mutual arithmetical relation are used to represent the geometric entities involved; subtraction and multiplication are thought of as counting procedures ("go up from X to Y"; "go X steps of Y"), and a square root is understood as the solution to the arithmetical equation $x \cdot x = A$.

Another change is found in the structure of the procedure. It is possible, and indeed plausible, that the procedure is still geometric - but in any case it is different from the Old Babylonian procedure. The latter would find the semi-difference between the length and width and would add it to
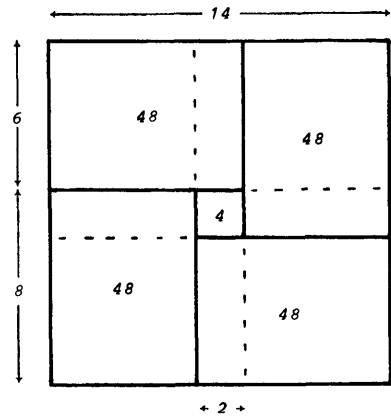
and subtract it from their semi-sum. Here, the *total* difference is found, and added to their sum, the result being then halved to yield the length, etc. The possible geometric argument is also made, so it looks, on a ready-made figure (see Figure 4) — the text contains no trace of constructive procedures.

IV. THE LIBER MENSURATIONUM

An 11th century (A.D.) manuscript contains a problem, which according to Cantor (1875, p. 104) may go back in its Latin version to the fourth century A.D., and which appears to have been translated from an Alexandrinian source. It deals with a right triangle, of which the hypotenuse and the area known. It leads to a second-degree equation, which is solved by the Seleucid method, — and indeed, the problem itself is closely related to the sort of problems dealt with in the Seleucid tablet just quoted. So, the Alexandrinian knowledge of second-degree equations appears to be more closely related to Seleucid than to Old Babylonian mathematics. This could lead to the idea that a continuous development goes from Old Babylonian texts over Seleucid and Alexandrinian applied mathematics to the earlier Middle Ages.

The more astonishing is the contents of a *Liber mensurationum*, "Book on ʿilm al-misâḥa", translated by Gherardo of Cremona in the 12th century A.D. from Arabic into Latin, and written originally by an otherwise unidentified Abû Bakr (cf. GAS V, 389f; Busard 1968 contains a critical edition).

The contents of the treatise is of evidently mixed origin. Its second half, dealing with trapezia, triangles, circle and circular sections, and finally with solids, has a strong Alexandrinian flavour. The first half (problems 1-64), dealing with square, rectangle, and rhomb, stands out for various reasons. It seems more archaic, and it is this part which I shall discuss here.

From various scattered references to "what precedes" it appears that the treatise was once companionpiece to a presentation of *al-jabr, aliabra* in the Latin text instead of the customary *algebra* (problems N° 5, 9, 25, 26 etc.). The translation appears to be very faithful, but at some points the text has been corrupted during the Arabic transmission (N° 38 refers to N° 32 as immediately preceding, and has furthermore taken up elements from another problem; N° 57, which is repeated as N° 61, refers to N° 58 as preceding; etc.).

The treatise is important both because of the way it is organized "rhetorically" and for its mathematical substance. To illustrate this I shall translate some of its problems ("Hinduizing" verbal numerals)

N° 3 If he (i.e. a "somebody" presented in N° 1) has said to you:
I have aggregated the side and the area (of a square), and what
resulted was 110. How much is then each of its sides?

The method of this will be that you take the half of its side as half
and multiply it with itself. 1/4 results, which you add to 110,
which will be 110 1/4. You then take the root of this, which is 10
1/2, from which you subtract the half, and 10 remain which are
the side. See!

There is also another method to it according to *al-jabr*, which is
that you take the side a thing and multiply it with itself, and what
results will be the wealth, which will be the area. Then add this to
the side as I said, and what results will be the wealth and a thing
which equals 110. Do then as it preceded for you in *al-jabr*,
which is that you halve the (coefficient of the) thing and multiply
it in itself, and what results you add to 110, and you take the root
of what comes out and subtract from it half the (coefficient of
the) root. What then remains will be the side.

N° 26 And if he has said to you: The area (of a rectangle) is 48, and
the longer side adds the quantity of 2 over the shorter side; what
then is each of the sides?

The method to find it will be that you halve the 2, and what results will be 1, which you multiply by itself, and 1 results. This
same you then join to 48, and 49 results, of which you take the
root which is 7, from which you subtract 1, and there remains 6
which is the shorter side. To this same then join 2, because his
speach was: one side exceeds the other by the quantity of 2, and
that which results will be 8. This then is the longer side.

But its method according to *al-jabr* is that you make the shorter
side a thing. Then the longer will be a thing and 2, multiply hence
a thing with a thing and with 2, whence wealth and 2 things will
equal 48, which is the area. Do then according to what preceded
for you in the fourth question (of *al-jabr*), and you will find it if it
pleases God.

N° 38 But if he has said to you: I have aggregated the longer and the shorter side and the area (of a rectangle), and what resulted was 62, while the longer side adds 2 over the shorter side; what is then each side?

The method to find this will be that you subtract 2 from 62, leaving back 60, and hence join 2 to half of the number of sides *(sic!),* from which 4 results. [...].

N° 45 But if he has said to you: I subtracted the longer side from the area (of a rectangle) and 40 remained, and the longer side adds 2 over the shorter side; what is then each side?
The method to find it will be, that you add 2 to 40, and it will be 42, which shall be kept in memory; then you subtract 1 from 2, and 1 remains. Take the half of it, which is 1/2, and multiply it with itself; and what results will be 1/4, which you shall join to the 42, and what results will be 42 1/2; take then its root, which is 6 1/2, and when the 1/2 is subtracted. 6 will remain which is the shorter side, over which the longer adds 2.

The method to find the same by *al-jabr* is simple.

Les us first look at the "rhetorical" aspect of the problems. The statements are formulated in the first person, preterite tense, by a "somebody". The same person and tense are used in the statement-part of Old Babylonian procedure texts,[4] and quite a few begin with the phrase *šumma kīam išāl-ka umma šū-ma,* "if somebody asks you thus:".[5] The beginning of the procedure-part, "the method to find it" etc., parallels the Old Babylonian *atta ina epēši-ka,* "you, by your method", and similar expressions; the ensuing shift to the second person, present tense, and to the imperative, is also a repetition of a fixed Old Babylonian pattern, — and so are the references back to the speaker of the statement in the third person.

More specifically, the construction of such references, "because his speach was" followed by a more or less literal quotation, corresponds to the

---

[4] But still, the excess of one side over the other is told in the present tense in Abû Bakr as well as in Old Babylonia!

[5] E.g. all the 11 problems published in Baqir 1951. Other texts carry the shorter *šumma,* "if", but subsequent references to the statement in the procedure-part of the problem show this word to be an ellipsis for the complete construction. Still others carry even no "if", but *all* have the statement in the first person preterite, as a teacher or a "somebody" telling what he has already done.

Old Babylonian *aššum iqbû*, "because he said", equally followed by a quotation. Finally, the "which shall be kept in memory" of N° 45 (and other problems) corresponds to a recurrent Old Babylonian *rēš-ka likil*, "may your head retain."

None of these features are found in Seleucid texts. Taken singly, each of them might be explained as a random coincidence. It is, however, extremely implausible that so many structural features should be repeated together randomly. Even though no texts of a similar structure are known from the span between the end of the Old Babylonian period, c. 1600 B.C., and the present work, we are forced to accept the existence of a continuous tradition during this immense span of time (and even of a *written* tradition, since purely oral transmission would hardly conserve the distinctions of tense and person in full sharpness). Furthermore, it appears that the Seleucid texts do not belong in the mainstream of this tradition.

One element of the rhetorical framework has no Old Babylonian counterpart, *viz.* the "See!" which closes N° 3 and many other procedure-descriptions of the treatise. The Latin word is *intellige*, "understand"/"see", but as the text stands it presents no appeal to the understanding - Gherardo offers only prescriptions, no explanation or justification. Two reasons suggest, however, that the original term was one involving visually supported understanding.

Firstly, another text translated by Gherardo describing an Indian way to construct equilateral polygons tells us that "they have in their hands no demonstration of this but the device: *Intellige ergo.*"[6] This can, however, only refer to the Indian way to close the description of a method by the word "See!" and a drawing.[7] So, in one place at least, Gherardo used *intelligere* as a (mis-) translation for an Arabic "See".

Secondly, the word is always to be found after the description of the basic procedure, the one which appears to descend directly from the naive-geometric Old Babylonian (cf. below); with procedures "according to *al-jabr*" it is strictly absent. Furthermore, an *intellige* in N° 2 corresponds to one of the few figures of the half of the treatise dealt with here. Finally, other figures belonging to the original treatise appear to have been lost in the process of transmission.

---

[6] The whole fragment is in Clagett 1984:600f.

[7] So in several of the texts and commentaries translated by Colebrooke (1817).

On the limit between rhetorics and mathematical substance we find the mathematical vocabulary. Here it is interesting that the square is spoken of as *quadratum equilaterum et orthogonium,* "equilateral and right-angled quadrate", while the rectangle is considered a quadratum altera parte longius, a "quadrate longer at one side." Evidently, the Arabic original was written in a context where the word normally translated in the twelfth century as *square (viz. murabba^c)* was still understood in its general, pre-theoretical sense of quadrangle (cf. also below, section VI). This usage is in itself a suggestion of rather archaic, sub-scientific roots for the main framework of the (first half of the) treatise.

If we now turn to real mathematical substance, three questions turn up: The choice and formulation of problems; the distinction between the normal, apparently unnamed method, and the methods of *al-jabr;* and the character of the normal method (or methods). The *al-jabr*-methods are those familiar from Al-Khwârizmî and other sources and give rise to no fundemental questions.

Concerning the choice of problems, it was already observed by Busard that a number of these (including occasionally the numbers involved) coincide with Old Babylonian or Seleucid problems (like most other authors, Busard does not distinguish the two). Since the number of simple second-depree algebraic problems and the number of e.g. simple pythagorean triples (important for the construction of problems on rectangles and rhombs) is restricted, I do not find this argument for direct connections very convincing — it would be utterly difficult to construct a statistical test of the hypothesis that the number of coincidences is greater than random.

What *can* be stated from problems alone is that the first half of the treatise is not just a *misâha*-handbook with the peculiarity that it makes use of algebraic methods: The majority of its problems would never occur in practical mensuration — instead, they can be obtained from such problems through interchange of known and unknown quantities; they are, in this sense, *algebraic* problems dressed in mensuration garments.

It can also be stated with great certainty that not all of the problems can derive from Babylonian sources. N° 51, dealing with a rectangle in which d:1::1:b and solving it apparently by reference to a division into extreme and mean ratio, could hardly have been formulated inside the conceptual framework of Babylonian mathematics. It seems related to early Greek (supposedly Pythagorean) geometry. The same may be the case of N[os] 16-17.

On the whole, however, the problems of the first part of the treatise are of a character reminding much of Old Babylonian and Seleucid mathematics, and which has little in common with Heronian and other Ancient material (and similar frail connection to Indian problem collections).

A particular feature of the text is the interest in the sum or difference between the area and the four sides of a square or a rectangle. It is represented by no less than six problems (N°s 4, 6, 9, 12, 43, and 46) . In earlier-mathematical traditions I know it only from the Old Babylonian BM 13901 N° 23 (cf. above), and from the possible reflection in Al-Khwârizmî's *Algebra (if* this is earlier).

When it comes to solutions, the most striking feature is that the first description of the "method to find it" is followed by a second "method according to *aliabra"* in many problems. Since both procedures can apparently be regarded with equal right (or equal lack of right) as algebraic in more modern senses of that word, *aliabra* (and hence *al-jabr)* must have a more restricted sense, to which Abû Bakr's counterposition can serve as a key.

In several cases (including N° 3, translated above) the numerical steps of basic and *al-jabr-* method are the same. The difference between the two must therefore be one of conceptualization or method, not one of algorithm (even though the algorithms are different in *most* cases). The explanation in N° 3 *(*and elsewhere) that the "wealth" is identical with the area shows us clearly that "wealth" and "root" are not to be understood *by themselves* as geometric quantities. *Al-jabr* is, according to the testimony of the text, concerned with the quantities "wealth", "root" and known number *connected arithmetically;* its problems are formulated and reduced to fundamental cases by arithmetico-rhetorical methods; and the fundamental cases are solved by automatic algorithms, involving no justification, proof or just conceptualization of the intermediate steps. This is in fact, if we disregard his naive-geometric justifications, precisely the *al-jabr* known from Al-Khwârizmî.

The basic method must then be *something different.* As the descriptions stand, it looks as if it appeals even less to any sort of understanding; still, whatever the meaning of *intellige,* be it "look" or "understand", this term involves *some* such appeal. Above, evidence speaking in favour of a visually supported understanding was discussed.

Further elucidation of the question may be achieved through investigation of N°s 38 and 45. We notice that the former is closely parallel to the Old Babylonian AO 8862 N° 1 (translated above, section II), the difference between the two amounting to a permutation of addition and subtraction. The reference to the "number of sides" shows that the text is mixed up with one of the problems dealing with a rectangle and its four sides (N°s 43 and 46), a corruption which is also clear from the ensuing numerical calculations (which is the reason why I have omitted the end of the problem). But already the beginning of the procedure shows that a shift of variable is intended, analogous to that of the Old Babylonian problem and reducing the problem to that of L · W=60, L—W=4 (L= 1+2). A similar reduction is performed in N° 45, where the whole procedure stands uncorrupted. It turns out to be precisely that of the Old Babylonian texts, using semi-sum and semi-difference.

This is a common feature of the first part of the *Liber mensurationum.* In contrast, the Seleucid standard method makes use of full sum and difference (see above, section III). This supports the impression coming from the rhetorical structure of the problems (and that given by "the four sides") that the first half of the *Liber mensurationum* is mainly affiliated directly to the Old Babylonian tradition, bypassing the Seleucid mathematicians, both regarding rhetorical and pedagogical build-up and as far as mathematical contents and method is concerned.

The surprising use of the term *quadratus* suggests that the translation is very consientious and literal. It should therefore be meaningful to submit Gherardo's text to precise terminological analysis in order to see to which extent the Old Babylonian conceptual distinctions are still conserved.

It turns out that the absolute distinctions between different multiplicatory operations ("making span", "raising" and "times") have been lost over the centuries. Still, there are a number of preferred modes of expression which agree well with with Old Babylonian ways. *Adjungare* (occasionally *addare)* for *waṣābum* "append", is one of them, and *aggregare* for *kamārum*, "accumulate", is another; quite a few others could be listed. This statistical but not always absolute dominance of certain terms in certain connections suggests that some variant of the old naive-geometric procedures was still in use, but that it was described verbally in a language the terminological structure of which was not (or was no longer) fully adapted to its concrete procedures.

Some of Abû Bakr's problems have no counterpart in published Old Babylonian texts but have so in the Seleucid tablet BM 34568 (notably N° 47). But the terminology used in even these problems carries precisely those features which were just described, and it is quite far from the complete arithmetization of the Seleucid tablet. So, Old Babylonian or not, these problems too appear to have developed inside the mainstream of the tradition leading from Old Babylonia to Abû Bakr, maybe before the Seleucid branch split off; they have in all probability *not* been borrowed from the outside in the way a few problems of Greek inspiration seem to have been taken over.

As explained above, the treatise shares the "See!" with many Indian texts. At the same time it is obvious that both problems and procedures differ from the sophisticated Indian syncopated algebra. Since the word recurs so frequently in the first part of the treatise but not in the "Alexandrinian" second part it is implausible that the usage can be a borrowing from India. Instead, it must belong with the mainstream development. As it is strictly absent from the Old Babylonian texts we can probably assume it to represent a change in the mainstream tradition taking place after Old Babylonian times.

All in all we may conclude that the first half of the *Liber mensurationum* represents a tradition which goes back Old Babylonian mathematics; which carries on the main features of the "rhetorical" structure of the Old Babylonian texts; and which was still making use of methods cognate to the naive geometry of the Babylonians when the Arabic original was formulated (but probbaly no longer when Gherardo made his translation). At the same time it presents us with an alternative, different, nongeometric tradition, identical in name and in contents with Al-Khwârizmîan *al-jabr*.

V. RELATIVES AND WITNESSES: SAVASORDA, FIBONACCI, THÂBIT AND ABÛ KÂMIL

Once the *Liber mensurationum* is known, it becomes obvious that Abraham bar Ḥiyya's (Savasorda's) *Collection on mensuration and partition (Ḥibbur ha-mᵉšîhah wᵉha-tišboret*, in *Latin Liber embadorum*, see the edition in Curtze 1902) is indebted to the same tradition for the part dealing with squares and rectangles (as both works depend on the Alexandrinian tradition for other parts). Since Abraham uses the same procedures as Abû Bakr and demonstrates their correctness in a geometric explanation followed by words like "and this is the figure" and a drawing, his treatise gives

us some support for the above interpretation of the word *intellige*. But Abraham draws directly on the *Elements* for his proofs instead of using naive manipulation of areas (the contents of II. 5 is quoted as trivial knowledge in Curtze 1902,40 [7ff], that of II.6 on p. 36 [10ff], and that of II. 7 on p. 42 [18ff]). Evidence from his hand can therefore only claim a hypothetical bearing on questions concerned with early Islamic, sub-scientific mathematical traditions.

The same can be said on Leonardo Fibonacci's *Practica geometriae,* which contains many of the same problems in the section on squares and rectangles (Boncompagni 1862, 56-77). Leonardo goes one step farther than Abraham in his syncrtetism, mixing up the old problems both with Euclidean principles and with the vocabulary of *al-jabr.*

The most important fact about these two run-away descendants of the tradition is that appear to be both mutually independent and independent of Abû Bakr. If so, the *Liber mensurationum* must be regarded as a representative of a wide-spread tradition in his times, not as a last survivor from a dying environment (cf. also on Abû Kâmil in the following section).

Alongside of this tradition, an old *al-jabr*-tradition must have existed. This can be seen in Thâbit's treatise "on the rectification of the cases of *al-jabr" (fî tashîh masâ᾽il al-jabr;* in Luckey 1941). Thâbit treats the subject through the three "elements" *(usûl)* of *al-jabr,* coincident with Al-Khwârizmî's 4th, 5th and 6th case but numbered from 1 to 3. The geometric proofs are also performed in (real or feigned) ignorance of Al-Khwârizmî's justifications. Further, the subject is labelled as stated, *not* as *al-jabr wa'l-muqâbala.* Finally, the subject is apparently not that of a book but one belonging with a group of practitioners, the *al-jabr*-people" *(ahl al-jabr)* or "followers of *al-jabr" (ashâb al-jabr).* If we think of the short span of time which separates Al-Khwârizmî and Thâbit (leaving no time for such a community to develop from scratch nor, *a fortiori,* to repress the memory of its founding father) it is clear that the company of *al-jabr* must be a group which was *not inspired by Al-Khwârizmî;* instead it *supplied him with inspiration.* (A further look at the text makes it clear that *al-jabr* as known to Thâbit is strictly identical with the discipline known to Abû Bakr under the same name).

In Abû Kâmil's *Algebra,* the idea of a special group of *al-jabr*-people seems to have disappeared. Instead, the subject, *is* now understood as the discipline of Al-Khwârizmî's *Kitâb fî al-jabr wa'l-muqâbala* (see the text in Levey 1966, 28f, including notes 1-2). There are, however, passages where

a plurality of distinct traditions are spoken of, namely problems N° 7 and 8 (Levey's counting). In N° 7 (Levey 1966, 92-95), the number 10 is to be divided into two parts, of which one is taken as the *thing* and the other as 10 minus the *thing;* this is well-known both from Al-Khwârizmî and from Abû Bakr's *al-jabr*-methods. *Alternatively,* the semi-difference between the two numbers is taken as the thing, and this way is referred to the "possessors of number" (b'ly h-mspr in the Hebrew text).

In N° 8, which also divides the number 10 into two parts (Levey 1966 94-103), it is the *al-jabr*-method (one number taken as the "thing") which is ascribed to a particular group, "those who pursue calculation" (ynhgw h-ḥšbnys). The closeness of Hebrew ḥšb and Arabic *ḥisâb* makes it fairly sure that Abû Kâmil spoke of people engaged in *ḥisâb,* practical commercial arithmetic, accounting etc. Astronomers or other scientific practitioners can hardly be meant.

These two references to groups of traditional sub-scientific mathematical practitioners are the only ones contained in Abû Kâmil's work, although he can be seen to draw on the methods of such environments in other places without indicating his source (see Anbouba 1978, 76, 82f). The subject is referred to Al-Khwârizmî, and it is given the full name of his presentation of the subject, *al-jabr wa'l-muqâbala.* At the same time the meaning of the term is widened, from the *al-jabr* of the *Liber mensurationum* to that of *algebra* in our sense. When Abû Kâmil was writing (early 4th/10th century?) the separate sub-scientific traditions were, at least when seen from Abû Kâmil's perspective, in the end of a process of absorption and integration with *mathematics* understood as a unified field ranging from high-level science to low-level but still reasoned and correct applications.[8] Even when considered as algebrists the mathematical practitioners of Islâm were becoming a "people of the Book", — and so, witnesses later than Abû Kâmil cannot be expected to have had access any longer to a situation similar to that encountered by Al-Khwârizmî and Ibn Turk who *wrote the Book.*

VI. AL-KHWÂRIZMÎ AND IBN TURK

Les us therefore return to these founding fathers, — first to Al-Khwârizmî, whose ample treatise offers more opportunity for analysis than the short fragment surviving from Ibn Turk.

---

[8] This general unification of Islamic mathematics and its cultural background is the main subject of my (1984). In reality the process was well under way but not nearly completed in the 4th / 10th century.

Al-Khwârizmî's starting point is *al-jabr, not* the basic method of the *Liber mensurationum*. This is clear already from his use of the "cases", from his use of the terms *mâl* ("wealth") and *jadr* ("root"), and from the subsequent arithmetico-rhetoric organization of the argument around the *šay‘* ("thing"). The Greek-tainted naive-geometric justifications are, already from their own formulation and appearance, grafted upon the main line of the book (and now when the existence of a naive-geometric tradition is certified we may assume with fair certainty that they were taken over from there). The secondary character of the geometric justifications is still more clear when the addition of.

(100 + wealth + 20 roots) and (50 + 10 roots + 2 wealths) is discussed (Rosen 1831, 33f). Here the author confesses that he has "contrived to construct a figure also for this case, but it was not sufficiently clear", while the "elucidation by words is very easy" and given rhetorically.

In the fragment of Ibn Turk's treatise the same basic orientation of thought in agreement with the al-jabr-pattern is also visible. Here too we have the standard cases, and here too they are defined in terms of *mâl* and *jadr*, not through the "area" and "side" which are the fundaments of the ensuing geometric justifications.

In Ibn Turk we find, however, a more outspoken parallel similarity with the naive-geometric tradition as reflected in the *Liber mensurationum* than in the case of Al-Khwârizmî. A square is indeed not simply a *murabba‘* to Ibn Turk but an "equilateral and equiangular *murabba‘*". The same usage is found only occasionally in Al-Khwârizmî, who in most places writes simply *murabba‘* (see Sayılı 1962, 84).[9]

Another similarity with the *Liber mensurationum* is more equally shared between the two. Both authors end their geometric explanations by a "This is the figure" (Al-Khwârizmî) or "And this is the shape of the Fi-

---

[9] This observation influences the question of priority and dependence. When Ibn Turk is so much closer than Al-Khwârizmî to the original use of a central term in the naive-geometric tradition, he can hardly have taken over his ideas from Al-Khwârizmî. Since the existence of two living traditions makes independent combination possible we cannot, on the other hand, conclude from here that Al-Khwârizmî copied Ibn Turk. Nor can we be sure that his writings are later. Most likely, the value of *murabba‘* was changing first in the circle of court mathematicias around Al-Ma'mûn, a place where the Greek influence was probably stronger than elsewhere. After all, the best literal translation of Greek τετράγωνον, "square", is nothing but *murabba‘*.

gure" (Ibn Turk), — preciely as it was also found in Abraham bar Ḥiyyâ's *Collection,* and corresponding to Abû Bakr's "see!".

So, we are led to the conclusion that both authors supplemented their treatise on the methods of the *"al-jabr*-people" with material borrowed from another sub-scientific tradition. They did so, however, from a conception of mathematics foreign to both sub-scientific traditions (as far as it can be judged from the indirect evidence at hand), namely from the idea that mathematics should be supplied with proofs.[10] This, and not only the use of letters to identify geometric entities and the way to explain the construction of a geometric figure, was in the scientific mathematical tradition initiated by the Greeks. The fundamental feat of the two authors was to bring the two levels of mathematical activity together for mutual fructification.

---

[10] It is precisely the lack of *explicit and autonomous* interest in proof (as distinct from practical and only implicit *understanding)* which makes me speak of *sub*-scientific traditions).

ABBREVIATIONS AND BIBLIOGRAPHY

Anbouba, Adel, 1978, "Acquisition de l'algèbre par les Arabes et premiers développ.ements. Aperçu général", *Journal for the History of Arabic Science,* vol. 2, pp. 66-100.

Baqir, Taha, 1951, "Some More Mathematical Texts from Tell Harmal", *Sumer,* vol. 7, pp. 28-45.

Boncompangi, Baldassare, ed., 1862, *Scritti di Leonardo Pisano,* II. Leonardi Pisani *Practica geometriae* ed *Opusculi,* Roma, Tipografia delle Scienze Matematiche e Fisiche.

Busard, H.L.L., 1968, "L'algèbre au moyen âge: Le 'Liber mensurationum' d'Abû Bekr", *Journal des Savants,* Avril-Juin 1968, pp. 65-125.

Cantor, Moritz, 1875, *Die römische Agrimensoren und ihre Stellung in der Geschichte der Feldmesskunst,* Leipzig, Teubner 1875.

Clagett, Marshall, 1984, *Archimedes in the Middle Ages,* V, *Quasi-Archimedean Geometry in the Thirteenth Century,* (Memoirs of the American Philosophical Society, 157 A+B), Philadelphia, The American Philosophical Society.

(Colebrooke, H.T.) 1817, *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhascara,* Translated by Henry Thomas Colebrooke, London: John Murray, Reprint: Wiesbaden, Martin Sändig, 1973.

Curtze, Maximilian, ed., 1902, *Urkunden zur Geschichte der Mathematik im Mittelalter and der Renaissance,* (Abhandlungen zur Geschichte der mathematischen Wissenschaften, 12-13), Leipzig, Teubner.

GAS: Fuat Sezgin, *Geschichte des arabischen Schrifttums,* I-IX (to date), Leiden, Brill, 1967.

Høyrup, Jens, 1982, "Oldbabylonisk algebra og geometrisk heuristik. En undersøgelse af en uventet begrebsverden", Roskilde Universitetscenter, Institut for Uddannelsesforskning, Medieforskining og Videnskabsteori, December 1982.

Høyrup, Jens, 1984, *Formative Conditions for the Development of Mathemacies in Medieval Islâm,* Contribution to the George Sarton Centennial, University of Ghent, 14-17 November 1984, Preliminary version, Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science.

Høyrup, Jens, 1984a, *Babylonian Algebra from the View-Point of Geomet-
    rical Heuristics, An Investigation of Terminology, Methods, and Pat-
    terns of Thought,* Roskilde University Centre, Institute of Educa-
    tional Research, Media Studies, and Theory of Science 1984.

Høyrup, Jens, 1985, *Babylonian Algebra from the View-Point. ...* Second,
    slightly corrected printing, Roskilde University Centre, Institute of
    Educational Research, Media Studies and Theory of Science.

Høyrup, Jens, 1985a, "Algebra and Naive Geometry. An Investigation of
    Some Basic Aspects of Old Babylonian Mathematical Thought",
    Manuscript in progress.

Levey, Martin, ed., 1966, The *Algebra* of Abû Kâmil, *Kitâb fî al-jabr wa'l-
    muqâbala,* in a Commentary by Mordechai Finzi, Hebrew Text,
    Translation, and Commentary with Special Reference to the Arabic
    Text, Madison, Milwaukee & London, University of Wisconsin
    Press 1966,

Luckey, Paul, 1941, "Ṯâbit b. Qurra über den geometrischen Richtigkeits-
    nachweis der Auflösung der quadratischen Gleichungen", *Sächsische
    Akademieder Wissenschaften zu Leipzig, Mathematisch-naturwis-
    senschaftliche Klasse, Berichte,* 93, pp. 93-114.

MKT. *Mathematische Keilschrift-Texte,* herausgegeben und bearbeitet von
    O, Neugebaure, I-III, (Quellen und Studien zur Geschichte der Ma-
    thematik,Astronomie und Physik, Abteilung A: Quellel, 3. Band),
    Berlin, Julius Springer 1935-1938.

Rosen, Frederic, ed., transl., 1831, The *Algebra* of Muhammed ben Musa,
    London, The Oriental Society.

Sayılı, Aydın, ed., transl., 1962, *Abdülhamid ibn-i Türk'ün Katışık Denk-
    lemlerde Mantıkî Zaruretler Adlı Yazısı ve Zamanın Cebri, (Logical
    Necessities in Mixed Equations by, ʿAbd al-Ḥamîd ibn Turk and the
    Algebra of His Time),* (Türk Tarih Kurumu Yayınlarından, VII. Seri,
    No. 41), Ankara, Türk Tarih Kurumu Basımevi.

TMB: F. Thureau-Dangin, ed., transl., *Textes mathématiques babyloniens,*
    (Ex Oriente Lux, Deel 1), Leiden, Brill 1938.

von Soden, Wolfram von. (Review, of E.M. Bruins & M. Rutten, *Textes
    mathématiques de Suse,* Paris 1961), *Bibliotheca Orientalis,* vol. 21,
    pp. 44-50.